Variational convergence of metric gradient flows

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Outline

1. Geodesically convex functions and EVI flows in metric spaces

2. Minimizing movements and uniform error estimates

3. Mosco convergence and stability results: the Hilbertian case

4. Stability in the metric setting

Motivation: gradient flows beyond Hilbert spaces

Classical setting: in a Hilbert space H, the gradient flow of a convex functional $\phi : H \to (-\infty, +\infty]$ is

$$\dot{u}(t) \in -\partial\phi(u(t)), \quad t > 0.$$

Key insight [Jordan-Kinderlehrer-Otto '98, Ambrosio-Gigli-Savaré '05]:
Many important PDEs can be interpreted as gradient flows in metric spaces without linear structure:

- the heat equation is the gradient flow of the Boltzmann entropy in the Wasserstein space of probability measures;
- the porous medium, Fokker-Planck, and other nonlinear diffusion equations fit the same framework.

Main question of this talk:

What happens when the energy functionals vary?

Can we deduce convergence of the flows from variational convergence of the energies?

The role of the EVI formulation

In a Hilbert space, convergence of gradient flows is equivalent to Mosco convergence of the energies [Attouch '84], and the proof relies on:

- resolvent operators $(I + \tau\partial\phi)^{-1}$ (contractions),
- the Crandall-Liggett exponential formula.

In a general metric space these tools are not available.
Our approach is based on:

1. the Evolution Variational Inequality (EVI) formulation of gradient flows,
2. quantitative error estimates for the minimizing movement (implicit Euler) scheme,
3. a suitable notion of variational convergence for the energies and the metric slopes.

For simplicity, we will present the results in the case $\lambda = 0$ (geodesic convexity without a lower curvature bound).

Geodesically convex functionals in metric spaces

(X, d) is a complete metric space.

$\phi : X \to (-\infty, +\infty]$ is a lower semicontinuous geodesically convex functional with non-empty domain $D(\phi) := \{x \in X : \phi(x) < \infty\}$.

Geodesically convex functions

ϕ is geodesically convex if for every $x_0, x_1 \in D(\phi) := \{x \in X : \phi(x) < \infty\}$ there exists a (minimal, constant speed) geodesic $\mathbf{x} : [0, 1] \to X$ such that

$$\phi(\mathbf{x}_\vartheta) \leqslant (1 - \vartheta)\phi(x_0) + \vartheta\phi(x_1) \quad \vartheta \in [0, 1].$$

Convexity along geodesics generalizes convexity in linear spaces: the graph of $\phi \circ \mathbf{x}$ lies below the chord joining $\phi(x_0)$ and $\phi(x_1)$.

The metric slope

The metric slope $|\partial\phi|$ is defined for $x \in D(\phi)$ by

$$|\partial\phi|(x) := \limsup_{y \to x} \frac{\big(\phi(x) - \phi(y)\big)_+}{\mathsf{d}(x, y)}$$

$|\partial\phi|(x) := +\infty$ if $x \in X \setminus D(\phi)$.

The metric slope is a lower semicontinuous map and plays the role of $\min\{|v| : v \in \partial\phi(x)\}$ in the Hilbert space case.

It measures the steepness of ϕ at x: a large slope means that ϕ decreases rapidly near x.

Evolution variational inequalities [Ambrosio-Gigli-S. '05]

A locally Lipschitz curve $\mathfrak{u} : (0, \infty) \to D(\phi)$ is a solution to the EVI(X, d, ϕ) if

$$\frac{1}{2}\frac{d}{dt}\mathsf{d}^2(\mathfrak{u}(t), v) \leqslant \phi(v) - \phi(\mathfrak{u}(t)) \quad \text{for a.e. } t > 0,\ v \in D(\phi).$$

Gradient flows

A *gradient flow* of ϕ is a family of continuous maps $\mathsf{S}_t : \overline{D(\phi)} \to \overline{D(\phi)}$, $t \geqslant 0$, such that for every $u_0 \in \overline{D(\phi)}$

$$\mathsf{S}_{t+h}(u_0) = \mathsf{S}_h(\mathsf{S}_t(u_0)) \quad \text{for every } t, h \geqslant 0, \qquad \lim_{t \downarrow 0} \mathsf{S}_t(u_0) = \mathsf{S}_0(u_0) = u_0,$$

the curve $t \mapsto \mathsf{S}_t(u_0)$ is a solution of EVI(X, d, ϕ).

Examples: Hilbert and Riemannian

$(X, |\cdot|)$ **is a Hilbert space**, $d(x, y) = |x - y|$, $x \mapsto \phi(x)$ is convex.

u a solution to EVI(X, d, ϕ) if and only if u is locally Lipschitz and

$$\dot{u}(t) \in -\partial\phi(u(t)) \quad t > 0,$$

where $v \in \partial\phi(x) \;\Leftrightarrow\; \langle v, y - x\rangle \leqslant \phi(y) - \phi(x)$ for every $y \in X$. In this case $|\partial\phi|(x) = \min\left\{|v| : v \in \partial\phi(x)\right\}$.

The **EVI formulation** follows from the differentiability of the squared distance:

$$\frac{1}{2}\frac{d}{dt}d^2(u(t), v) = \langle \dot{u}(t), u(t) - v\rangle = \langle -\dot{u}(t), v - u(t)\rangle \leqslant \phi(v) - \phi(u(t))$$

Similarly on a complete Riemannian manifold (X, g): if $\phi \in C^2(X)$ with $\operatorname{Hess}\phi \geqslant 0\, g$, then u solves EVI iff $\dot{u}(t) = -\nabla\phi(u(t))$, $t > 0$.

Let $(X, \mathsf{d}, \mathfrak{m})$ **be a complete metric measure space**, $\mathfrak{m} \in \mathscr{P}(X)$. Let $\mathfrak{X} := \mathscr{P}_2(X)$ endowed with the Wasserstein distance W_{d}.

The gradient flow of the entropy in $\mathfrak{X}$ is the heat flow. Let $\phi(\mu) := \mathrm{Ent}(\mu|\mathfrak{m})$,

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int \rho \log \rho \, d\mathfrak{m} \quad \text{if } \mu = \rho\mathfrak{m}; \qquad +\infty \text{ if } \mu \not\ll \mathfrak{m}.$$

$(X, \mathsf{d}, \mathfrak{m})$ **is** $\mathrm{RCD}(0, \infty)$ **iff** $\mathrm{Ent}(\cdot|\mathfrak{m})$ **admits a gradient flow in** $(\mathscr{P}_2(X), W_{\mathsf{d}})$.

In this case $\mathsf{S}_t\mu = \mathsf{u}(t, \cdot)\mathfrak{m}$ where u solves

$$\partial_t \mathsf{u} = \Delta \mathsf{u} \quad t > 0; \qquad \lim_{t \downarrow 0} W_{\mathsf{d}}(\mathsf{u}(t, \cdot)\mathfrak{m}, \mu) = 0.$$

Main properties of solutions to EVI

Let $\mathbf{u}^1, \mathbf{u}^2 \in C^0([0,+\infty);X)$ be solutions of $\mathrm{EVI}(X, \mathsf{d}, \phi)$.

- **Contraction and uniqueness:**

$$\mathsf{d}(\mathbf{u}^1(t), \mathbf{u}^2(t)) \leqslant \mathsf{d}(\mathbf{u}^1(0), \mathbf{u}^2(0)) \quad \text{for every } t > 0.$$

  In particular, for every $u_0 \in \overline{D(\phi)}$ there is at most one solution $\mathbf{u}$ of $\mathrm{EVI}(X, \mathsf{d}, \phi)$ satisfying the initial condition $\lim_{t \downarrow 0} \mathbf{u}(t) = u_0$.

- **Regularizing effect:** $\mathbf{u}$ is *locally Lipschitz* in $(0,+\infty)$ and the map $t \mapsto \phi(\mathbf{u}(t))$ is *nonincreasing*. Moreover, the energy identity holds:

$$\frac{d}{dt}\phi(\mathbf{u}(t+)) = -|\dot{\mathbf{u}}(t+)|^2 = -|\partial\phi|^2(\mathbf{u}(t)) \quad \text{for every } t > 0.$$

Minimizing movements

Given $\tau > 0$ and $U_\tau^0 \approx u_0$, consider $\Phi(\tau, U, V) := \frac{1}{2\tau}\mathbf{d}^2(U, V) + \phi(V)$.

$(U_\tau^n)_n$ **is a discrete minimizing sequence** if $U_\tau^n \in \operatorname{Argmin}\Phi(\tau, U_\tau^{n-1}, \cdot)$, i.e.

$$\frac{1}{2\tau}\mathbf{d}^2(U_\tau^{n-1}, U_\tau^n) + \phi(U_\tau^n) \leqslant \frac{1}{2\tau}\mathbf{d}^2(U_\tau^{n-1}, V) + \phi(V) \quad \text{for every } V \in X.$$

The **discrete minimizing movement** is the piecewise constant interpolant $\mathbf{U}_\tau$:

$$\mathbf{U}_\tau(t) := U_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau].$$

Minimizing Movements

$\mathbf{u}$ is a **Minimizing Movement** in $MM(\Phi, u_0)$ if there exists $(\mathbf{U}_\tau)_\tau$ such that

$$\mathbf{u}(t) = \lim_{\tau\downarrow 0}\mathbf{U}_\tau(t) \quad \text{for every } t > 0.$$

Ekeland relaxation of minimizing movements

Consider a relaxation parameter $\eta \in (0,1)$.

Applying **Ekeland's variational principle** to $V \mapsto \Phi(\tau, U_\tau^{n-1}, V)$, starting from $W := U_\tau^{n-1}$ we find U_τ^n satisfying

$$\frac{1}{2\tau}\mathsf{d}^2(U_\tau^{n-1}, U_\tau^n) + \phi(U_\tau^n) \leqslant \frac{1}{2\tau}\mathsf{d}^2(U_\tau^{n-1}, V) + \phi(V) + \eta\, \mathsf{d}(U_\tau^{n-1}, U_\tau^n)\mathsf{d}(U_\tau^n, V)$$

for every $V \in X$, and

$$\frac{1}{2\tau}\mathsf{d}^2(U_\tau^{n-1}, U_\tau^n) + \phi(U_\tau^n) \leqslant \phi(U_\tau^{n-1}) .$$

For $\eta = 0$ we recover the usual Minimizing Movement scheme.

We call $\mathbf{U}_{\tau,\eta}$ any **piecewise constant interpolant** of a discrete η-Ekeland-minimizing sequence.

Resolvent and Crandall-Liggett estimates in the Hilbert case

In a Hilbert space, $\mathbf{A} := \partial\phi$. The Minimizing Movement scheme is the backward Euler method, solved by the **resolvent** $\mathbf{J}_\tau := (I + \tau\mathbf{A})^{-1}$:

$$\frac{U_\tau^n - U_\tau^{n-1}}{\tau} + \mathbf{A}U_\tau^n \ni 0, \qquad U_\tau^n = \mathbf{J}_\tau U_\tau^{n-1}.$$

$\mathbf{J}_\tau$ is a contraction $\Rightarrow$ Crandall-Liggett exponential formula:

$$\mathbf{u}(t) = \lim_{n\to\infty} \mathbf{J}_{t/n}^n u_0, \qquad |\mathbf{u}(t) - \mathbf{U}_\tau(t)| \leqslant 2|\mathbf{A}(u_0)|\sqrt{T\tau} \quad t \in [0, T].$$

The Ekeland minimization provides a solution of

$$\frac{U_\tau^n - U_\tau^{n-1}}{\tau} + \mathbf{A}U_\tau^n \ni \Xi_\tau^n, \quad |\Xi_\tau^n| \leqslant \eta\, |U_\tau^n - U_\tau^{n-1}|.$$

Theorem (A priori error estimates)

Suppose that ϕ admits a gradient flow and fix a time interval $T > 0$. There exists a constant $C = C(T)$ such that every discrete η-Ekeland minimizing movement $\mathbf{U}_{\tau,\eta}$ and every solution $\mathbf{u}$ of the $\mathsf{EVI}(X, \mathsf{d}, \phi)$ *satisfy*

$$\mathsf{d}(\mathbf{u}(t), \mathbf{U}_{\tau,\eta}(t)) \leqslant \mathsf{d}(u_0, U^0_{\tau,\eta}) + C|\partial\phi|(U^0_{\tau,\eta})\sqrt{\tau}.$$

In particular, if $U^0_{\tau,\eta} = u_0$,

$$\mathsf{d}(\mathbf{u}(t), \mathbf{U}_{\tau,\eta}(t)) \leqslant C|\partial\phi|(u_0)\sqrt{\tau} \quad t \in [0, T].$$

Corollary (Minimizing Movements coincide with gradient flows)

If ϕ admits a gradient flow then $\mathrm{MM}(\Phi, u_0)$ coincides with the solution $\mathbf{u}(t) = \mathbf{S}_t u_0$ of the $\mathsf{EVI}(X, \mathsf{d}, \phi)$ *starting from u_0.*

The stability problem: given ϕ^h geodesically convex with gradient flows S^h, and $\phi^h \to \phi$ in a suitable sense, does $S^h_t u \to S_t u$?

In a Hilbert space we can use the **weak and strong topology**.
ϕ^h **Mosco-converges to** ϕ if

$$x^h \rightharpoonup x \quad \Rightarrow \quad \liminf_{h\to\infty} \phi^h(x^h) \geqslant \phi(x)$$
$$\forall x \in X \quad \exists (x^h)_h : \qquad x^h \to x, \quad \phi^h(x^h) \to \phi(x).$$

Equivalently: ϕ^h Γ-converges to ϕ w.r.t. both the weak and the strong topology:

$$\phi = M\text{-}\lim_{h\to\infty} \phi^h \quad \Longleftrightarrow \quad w\text{-}\Gamma\text{-}\liminf_{h\to\infty} \phi^h = s\text{-}\Gamma\text{-}\limsup_{h\to\infty} \phi^h = \phi.$$

The Hilbert case

Suppose that X is a Hilbert space, $(\phi^h)_{h\in\mathbb{N}}$, ϕ l.s.c. and convex.

Theorem (Benilan, Crandall, Pazy, Attouch)

The following properties are equivalent:

- **Convergence of the flows:** *whenever* $u_0^h \to u_0 \in D(\phi)$, $u_0^h \in D(\phi^h)$, *we have*

$$\lim_{h\to\infty} \mathsf{S}_t^h u_0^h = \mathsf{S}_t u_0 \quad \textit{for every } t \geqslant 0.$$

- **Mosco-convergence:** $\phi^h \xrightarrow{M} \phi$.
- **Convergence of the resolvents:** $(I + \tau\mathbf{A}^h)^{-1}u_0 \to (I + \tau\mathbf{A})^{-1}u_0 \ \forall u_0 \in X$, $\tau > 0$.
- **Moreau-Yosida regularizations:** *for every* $u \in X$, $\tau > 0$*:*

$$\inf_v \phi^h(v) + \frac{1}{2\tau}\mathsf{d}^2(v,u) \to \inf_v \phi(v) + \frac{1}{2\tau}\mathsf{d}^2(v,u)$$

- G**-convergence of** A^h**:** $\forall v \in \mathbf{A}(u)\ \exists u^h, v^h$: $v^h \in \mathbf{A}^h u^h$, $u^h \to u$, $v^h \to v$.

Remarks

- Mosco limits of convex functionals are convex: the limit semigroup $\mathcal{S}$ exists thanks to Crandall-Liggett.
  In metric spaces, the existence of the limit flow is not guaranteed a priori.
- Bounded sequences $(x^h)_h$ are always weakly compact in Hilbert spaces; without strong coercivity, one must deal with limits of $\phi^h(x^h)$ along weakly convergent sequences, so that the weak Γ-$\liminf$ is involved.
  In metric spaces, a "weak" topology is generally missing.

Other references and settings

- [Mosco '67-'69] obstacle problems; [Spagnolo '67, De Giorgi-Spagnolo '73] heat flow, Γ-convergence.
- [S.-Visintin '97, Ambrosio-S.-Zambotti '09, Peletier-S.-Veneroni '10]: **energies and metrics depend on** h.
- [Ambrosio-Gigli-S. '05-14, Gigli-Mondino-S. '15]: **stability of RCD spaces**.
- [Ortner '05, Serfaty '11]: slopes, non-convex setting. [Visintin '14-'17]: **Fitzpatrick** approach.
- [Mielke, Roubicek, Stefanelli; Mielke-Rossi-S.]: **rate-independent** problems.

Stability via error estimates

In the Hilbert space setting, stability follows by Crandall-Ligget estimate. Suppose that $u_0^h \to u_0$, $\sup_h |\mathbf{A}^h(u_0^h)| < \infty$, $t \in [0,T]$ and $\tau = T/N$.

- First of all we replace the family $\mathbf{u}^h(t) = \mathbf{S}_t^h(u_0^h)$ with the discrete minimizing movement $\mathbf{U}_\tau^h$ starting from u_0^h: we pay an error $\mathbf{d}(\mathbf{u}^h(t), \mathbf{U}_\tau^h(t)) \leqslant \varepsilon_\tau = C\sqrt{\tau}$, $t \in [0,T]$, independent of h.
- Similarly, we replace the limit solution $\mathbf{u}(t) = \mathbf{S}_t(u_0)$ with the discrete minimizing movement $\mathbf{U}_\tau$ starting from u_0 and we pay an error $\mathbf{d}(\mathbf{u}(t), \mathbf{U}_\tau(t)) \leqslant \varepsilon_\tau = C\sqrt{\tau}$, $t \in [0,T]$.
- By triangle inequality
$$\mathbf{d}(\mathbf{u}^h(t), \mathbf{u}(t)) \leqslant \mathbf{d}(\mathbf{U}_\tau^h(t), \mathbf{U}_\tau(t)) + 2C\sqrt{\tau}$$
- We fix $\varepsilon > 0$ and choose τ sufficiently small, so that $2C\sqrt{\tau} \leqslant \varepsilon/2$. $\mathbf{U}_\tau^h$ and $\mathbf{U}_\tau$ depend on a fixed number of time steps $N := T/\tau$, and thus of iterations of the resolvent operators.
  The convergence of the resolvent implies that for $h \geqslant \bar{h}$ sufficiently big
$$\mathbf{d}(\mathbf{U}_\tau^h(t), \mathbf{U}_\tau(t)) \leqslant \varepsilon/2.$$
  We deduce
$$\mathbf{d}(\mathbf{u}^h(t), \mathbf{u}(t)) \leqslant \varepsilon \quad \text{if } h \geqslant \bar{h}.$$

Difficulties of the metric setting

- We do not know a priori that the limit flow S exists.
- Resolvent are not well defined and we should use the relaxed Ekeland version.
- Even if resolvents are well defined, they are not contractions.
- A "weak" topology is missing in general.
- We would like to study stability without assuming strong coercivity of the functionals.
- Minimizers (and, even worse, Ekeland minimizers) do not behave well under Γ-convergence without coercivity.

Main stability result: equivalent conditions (I)

Let ϕ^h, ϕ be l.s.c. and geodesically convex.

Theorem (Muratori-S.)

The following properties are equivalent:

1. **Convergence of the flows:** *whenever* $u_0^h \to u_0 \in D(\phi)$, $u_0^h \in D(\phi^h)$,

$$\lim_{h\to\infty} \mathsf{S}^h(t)u_0^h = \mathsf{S}(t)u_0 \quad \textit{for every } t \geqslant 0.$$

2. **Recovery sequence for ϕ and $|\partial\phi|$:** *for every* $u \in D(|\partial\phi|)$ *there exist* $u^h \in D(|\partial\phi^h|)$ *such that*

$$u^h \to u, \quad \phi^h(u^h) \to \phi(u), \quad \limsup_{h\to\infty} |\partial\phi^h|(u^h) \leqslant |\partial\phi|(u).$$

3. **Γ-convergence for ϕ^h and $|\partial\phi^h|$:** $\Gamma\text{-}\lim_{h\to\infty} \phi^h = \phi, \quad \Gamma\text{-}\lim_{h\to\infty} |\partial\phi^h| = |\partial\phi|.$

Condition 2 is the metric counterpart of the Hilbertian recovery condition $u^h \to u$, $v^h \to v$, $v^h \in \mathbf{A}^h(u^h)$, $\phi^h(u^h) \to \phi(u)$.

Main stability result: equivalent conditions (II)

Theorem (continued)

The previous conditions are also equivalent to:

4. **Qualified Γ-convergence:** $\Gamma\text{-}\limsup_{h\to\infty} \phi^h \leqslant \phi$ *and for every* $u \in D(\phi)$, $\varepsilon > 0$ *and* $\bar{\tau} > 0$, *there exists* $\tau \in (0, \bar{\tau})$ *such that*

$$\liminf_{h\to\infty} \inf_{B_\tau(u)} \phi^h \geqslant \inf_{B_\tau(u)} \phi - \varepsilon\tau$$

5. **Local Yosida regularization:** $\Gamma\text{-}\limsup_{h\to\infty} \phi^h \leqslant \phi$ *and for every* $u \in D(\phi)$, $\varepsilon > 0$ *and* $\bar{\tau} > 0$, *there exists* $\tau \in (0, \bar{\tau})$ *such that*

$$\liminf_{h\to\infty} \inf_{v} \phi^h(v) + \frac{1}{2\tau}\mathbf{d}^2(u, v) \geqslant \inf_{v} \phi(v) + \frac{1}{2\tau}\mathbf{d}^2(u, v) - \varepsilon\tau$$

Conditions 4 and 5 are purely variational and do not involve the metric slope: they provide a quantitative reinforcement of Γ-convergence that controls the local behavior of the infima.

Mosco convergence in metric spaces

Suppose that X is endowed with a weaker topology σ such that

- $x_n \xrightarrow{\sigma} x$ yields $\liminf_{n\to\infty} \mathbf{d}(x_n, y) \geqslant \mathbf{d}(x, y)$ for every $y \in X$
- every $(x^h)_h$ bounded sequence with $\phi^h(x^h) \leqslant C$ admits a σ-convergence subsequence.

We suppose that ϕ^h Mosco converges to ϕ in $(X, \mathbf{d}, \sigma)$:

$$
\begin{aligned}
x^h \xrightarrow{\sigma} x \quad &\Rightarrow \quad \liminf_{h\to\infty} \phi^h(x^h) \geqslant \phi(x) \\
\forall\, x \in X \quad \exists\, (x^h)_h : \quad & \quad x^h \to x, \quad \phi^h(x^h) \to \phi(x).
\end{aligned}
$$

Theorem

If ϕ^h admits a gradient flow S^h then also ϕ admits a gradient flow and $\mathsf{S}^h_t u^h_0 \to \mathsf{S}_t u_0$ for every $u^h_0 \in \overline{D(\phi^h)}$ converging to u_0.

Two basic estimates ($\eta = 0$)

(Ekeland) minimizers satisfy a discrete energy inequality:

$$\tau|\partial\phi|^2(U_\tau^n) \leqslant \frac{\mathbf{d}^2(U_\tau^{n-1}, U_\tau^n)}{\tau} \leqslant \phi(U_\tau^{n-1}) - \phi(U_\tau^n) \qquad (\star)$$

Key point: if $(\star)$ holds up to an error $\varepsilon > 0$ at the level h, one still gets

$$\mathbf{d}(\mathbf{u}^h(t), \mathbf{U}_\tau^h(t)) \leqslant \mathbf{d}(u_0, U_\tau^{0,h}) + C\left(\sqrt{\tau} + \sqrt{\varepsilon}\right)$$

provided $u_0 \in D(|\partial\phi|)$ and $\sup_{h,\tau} |\partial\phi|(U_\tau^{0,h}) < \infty$.

Strategy

- Generate a sequence of (Ekeland) minimizers for ϕ satisfying

$$\tau|\partial\phi|^2(U_\tau^n) \leqslant \frac{\mathbf{d}^2(U_\tau^{n-1}, U_\tau^n)}{\tau} \leqslant \phi(U_\tau^{n-1}) - \phi(U_\tau^n), \tag{$\star$}$$

- Use Γ convergence of ϕ^h and $|\partial\phi^h|$ to approximate U_τ^n by sequences $U_\tau^{n,h}$ satisfying $(\star)$ up to a small error $\varepsilon > 0$ with

$$\lim_{h\to\infty} \sup_{t\in[0,T]} \mathbf{d}(\mathbf{U}_\tau(t), \mathbf{U}_\tau^h(t)) = 0.$$

- Use the a priori error estimate to prove that

$$\mathbf{d}(\mathbf{u}^h(t), \mathbf{U}_\tau^h(t)) \leqslant \mathbf{d}(u_0, U_\tau^{0,h}) + C\left(\sqrt{\tau} + \sqrt{\varepsilon}\right)$$

- Combining the two estimates, we deduce

$$\limsup_{h,k\to\infty} \mathbf{d}(\mathbf{u}^h(t), \mathbf{u}^k(t)) \leqslant 2C\left(\sqrt{\tau} + \sqrt{\varepsilon}\right)$$

which shows that $(\mathbf{u}^h(t))_h$ is a Cauchy sequence, since $\tau > 0$ and $\varepsilon > 0$ are arbitrary.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, \infty)$ metric measure space and let $\varphi : X \to (-\infty, +\infty]$ be a continuous and geodesically convex function.

Theorem (Sturm '14)

If (X, d) is locally compact then φ admits a gradient flow.

Corollary

The local compactness assumption can be removed.

The proof relies on the construction of the gradient flow for the functional

$$\phi(\mu) := \int_X \varphi \, d\mu \quad \text{in } (\mathscr{P}_2(X), W_{\mathsf{d}})$$

by considering the approximation $\phi^h(\mu) := \phi(\mu) + \frac{1}{h}\,\mathrm{Ent}(\mu|\mathfrak{m})$. At least when $\mathfrak{m} \in \mathscr{P}(X)$ it is easy to check that the assumptions of the main stability result are satisfied.

Main contributions:

- Quantitative error estimates for the minimizing movement scheme in metric spaces, uniform w.r.t. approximating functionals.
- A complete characterization of variational convergence of gradient flows: five equivalent conditions, from convergence of flows to qualified Γ-convergence.
- No need for strong coercivity or linear structure: the results apply to Wasserstein gradient flows and RCD spaces.

Further results (not discussed here for simplicity):

- The full λ-convex and approximately convex settings.
- Stability under Gromov-Hausdorff convergence of the underlying metric spaces.

Open: convergence rates for the flows in terms of the approximation parameters.

Thank you!